

# ON SOME ASPECTS OF THE GEOMETRY OF DIFFERENTIAL EQUATIONS IN PHYSICS

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## Abstract

In this review paper, we consider three kinds of systems of differential equations, which are relevant in physics, control theory and other applications in engineering and applied mathematics; namely: Hamilton equations, singular differential equations, and partial differential equations in field theories. The geometric structures underlying these systems are presented and commented. The main results concerning these structures are stated and discussed, as well as their influence on the study of the differential equations with which they are related. Furthermore, research to be developed in these areas is also commented.

**Key words:** *Hamilton equations, singular systems, field theories, symplectic and presymplectic geometry, multisymplectic manifolds.*

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## 1 Introduction

The aim of this paper is to make evident the relation between some kinds of systems of differential equations and certain underlying (hidden) geometric structures, whose analysis may allow us to clarify the properties of the system described by those equations.

We pay attention to three different situations. Two of them concern ordinary differential equations (ODEs), and the third one is partial differential equations (PDEs).

The first one is (autonomous) Hamiltonian systems, with the symplectic geometry as background structure. In fact, there is a historical interplay between the equations and the geometric structure, and results from one of these aspects depend for their interpretation on the other, leading to new insights in both aspects. This subject is treated in Section 2.

The second is singular differential equations; that is, those which cannot be written in normal form; and in particular, the case where the dependence on the derivatives is linear. The underlying structure is a submanifold of a tangent bundle. The study of systems of singular differential equations was made separately in theoretical physics and in some technical areas such as engineering of electric networks or control theory. In both cases, algorithms for solving the problems (which are essentially the same) were developed, although independently. The study of the underlying geometric structure has proved to be invaluable to find the solutions of these equations and understand their properties. All of this is developed in Section 3.

Finally, the third subject presents a different situation, and concerns partial differential equations appearing in physics and technical applications. Many partial differential equations admit a variational formulation. Multisymplectic geometry underlies this for-

mulation. It seems to be a generalization of symplectic geometry, but the difficulty of the problems is of increasing magnitude. One of the objectives of this lecture is to present some of these problems, enabling us to clarify some aspects related to all these partial differential equations. Section 4 is devoted to it.

Manifolds are supposed to be real, paracompact, and  $C^\infty$ . Maps are  $C^\infty$ . Sum over crossed repeated indices is understood.

## 2 Hamilton equations and symplectic geometry

### 2.1 Hamilton equations

The study of (time-independent) Hamiltonian systems is a classical subject which arises from the transformation of the Euler-Lagrange equations, a second-order system of ODEs, into a first-order system, using the so-called *Legendre transformation* [1], [2], [100].

For a system defined by a Lagrangian function  $L(t, q^i, \dot{q}^i)$ , the Euler-Lagrange equations are

$$[L]_i := \frac{\partial L}{\partial q^i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) = 0, \quad i = 1, \dots, m \quad (1)$$

which is a second-order system for  $(q^1, \dots, q^m)$ . The associated Hamiltonian system introduces new variables  $(p_1, \dots, p_m)$  called *momentum coordinates*, and the so-called *Hamiltonian function*, defined as  $H(t, q^i, p_i) = \dot{q}^i p_i - L(t, q^i, \dot{q}^i)$ , where  $(\dot{q}^1, \dots, \dot{q}^m)$  are supposed to be solved from the relations

$$p_i = \frac{\partial L}{\partial \dot{q}^i}, \quad i = 1, \dots, m$$

With all these data, the Hamilton equations (Lagrange, 1808) are

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad i = 1, \dots, m \quad (2)$$

It is well known that both systems (1) and (2) are equivalent in the following sense: if  $(q^1(t), \dots, q^m(t))$  is a solution of (1), then  $(q^1(t), \dots, q^m(t); p_1(t), \dots, p_m(t))$ , with  $p_i(t) = \frac{\partial L}{\partial \dot{q}^i}(t, q^i(t), \dot{q}^i(t))$ , is a solution of (2), and conversely. The procedure of transforming Euler-Lagrange's equations into Hamilton's equations is known as *Legendre transformation*.

This situation is the usual for classical systems. Nevertheless, there is a large class of interesting physical models (mainly related with relativistic systems) for which this procedure is not possible. They are called *singular* systems, and will be discussed in Section 3.

It is important to point out that there are also dynamical systems of Hamiltonian type (i.e., whose dynamical equations are of the form (2)) which have no Lagrangian counterpart; that is, they are not defined by a Lagrangian function, but by giving a Hamiltonian function (see, for instance, [91]).

## 2.2 Geometric formulation of Hamilton's equations

At this point we can ask about the geometric structure underlying these systems of ODEs. For simplicity we can only consider the time-independent case. In the first case, we have the tangent bundle  $TQ$  of the manifold  $Q$  which represents the configuration space of the system. The tangent bundle has some natural geometric structures. Among them, the vertical endomorphism and the Liouville vector field can be used to construct several geometric objects that allow us to express the Euler-Lagrange equations in an intrinsic form. Nevertheless, it seems that the geometric structure underlying the Hamilton equations is more interesting for many applications. In order to obtain it, observe that these equations can be written in matrix form

$$(\dot{q}^i, \dot{p}_i) \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{pmatrix} = \left( \frac{\partial H}{\partial q^i}, \frac{\partial H}{\partial p_i} \right)$$

The skew-symmetric matrix  $\omega = \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{pmatrix}$  can be interpreted as a 2-form  $\omega = dq^i \wedge dp_i$ , and the equation means that the contraction of  $\omega$  with the velocity vector  $\dot{q}^i \frac{\partial}{\partial q^i} + \dot{p}_i \frac{\partial}{\partial p_i}$  equals the 1-form  $dH = \frac{\partial H}{\partial q^i} dq^i + \frac{\partial H}{\partial p_i} dp_i$ . In intrinsic terms,  $(q^i(t), p_i(t))$  is the local representation of a path  $\xi$  in a manifold  $M$  (the phase space of the system), endowed with a 2-form  $\omega$ , and  $\xi$  satisfies the differential equation

$$i_\xi \omega = dH \circ \xi$$

where  $H: M \rightarrow \mathbb{R}$  is a function. Furthermore,  $\omega$  is required to be symplectic; that is, closed and non-degenerate. So we reach the concept of *symplectic manifold*  $(M, \omega)$ , which is a geometric structure similar to that of a Riemannian manifold. In a symplectic manifold, given a function  $f \in C^\infty(M)$ , there exists a unique vector field  $X_f \in \mathfrak{X}(M)$  associated with  $f$ , such that  $i_{X_f} \omega = df$ .  $X_f$  is called the *Hamiltonian vector field* associated with  $f$ , and  $f$  is called the *Hamiltonian function* associated with  $X_f$ . Hence, note that the Hamilton equations for  $\xi$  can be written as  $\dot{\xi} = X_H \circ \xi$ , where  $X_H$  is the Hamiltonian vector field for  $H$ .

Elementary examples of symplectic manifolds are  $\mathbb{R}^2$  and  $S^2$ , both with the area element  $\omega$ , and the cotangent bundle  $T^*Q$  of the configuration manifold  $Q$  of the system, which carries a canonical symplectic structure. In this last case, the above mentioned Legendre transformation is a map  $\mathcal{FL}: TQ \rightarrow T^*Q$ .

As we will see in Section 3, this geometric framework can be extended to describe singular Hamiltonian systems, just allowing the closed form  $\omega$  to be degenerate (then it is called a presymplectic form, and  $(M, \omega)$  is said to be a *presymplectic manifold*).

An skew-symmetric bilinear map on  $C^\infty(M)$ , which is called the *Poisson bracket*, can be defined in every symplectic manifold  $(M, \omega)$  as follows

$$\{f, g\} := \omega(X_g, X_f)$$

which allows us to write the Hamilton equations in the form

$$\dot{q}^i = \{q^i, H\}, \quad \dot{p}_i = \{p_i, H\}$$

(Actually, the concept of Poisson bracket is more general than the symplectic structure [59], [65]).

A relevant feature of Hamiltonian vector fields is that they preserve the symplectic (resp. presymplectic) form. In fact,  $X \in \mathfrak{X}(M)$  is a (locally) Hamiltonian vector field iff  $L_X\omega = 0$ . In the simple example of the symplectic manifold  $(\mathbb{R}^2, \omega)$ , this result means that the area of a region  $U \subset \mathbb{R}^2$  and the area of the transformed region  $\varphi(U) \subset \mathbb{R}^2$ , under the flow of a Hamiltonian vector field  $X$ , are the same. This flow is an example of a map preserving the symplectic form, that is, a symplectomorphism.

## 2.3 Main results concerning symplectic geometry

In 1912, Poincaré, working on the symplectic manifold  $(\mathbb{R}^2, \omega)$ , proposed a simplified 2-dimensional model for the solar system, obtaining the first result on symplectic geometry which was used for studying Hamiltonian systems (see [2], [9]):

**Theorem 1** (Poincaré’s last geometric theorem). *Suppose  $\varphi: A \rightarrow A$  is an area-preserving diffeomorphism of the closed annulus  $A = \mathbb{R}/\mathbb{Z} \times [-1, 1]$ , which preserves the two components of the boundary, and twists them in opposite directions. Then  $\varphi$  has at least two fixed points.*

This result leads to the concept of Poincaré map for Hamiltonian systems, and to determining the existence of periodic orbits for many dynamical systems. There is a generalization of this result (the so-called *Arnold’s conjecture*) and several applications of it ([9], pp. 55-56). In particular, the relation between the number of fixed points of a symplectomorphism and the critical points of a *Morse function* in the manifold [2], [72].

This was the standpoint of a long sequence of results on the geometry and topology of symplectic manifolds, whose developments have led to a clarification of the structure and properties of Hamiltonian systems. Some of them are the following (see, for instance, [1], [59], [72], [99] for details):

**Theorem 2** (Darboux, 1882). *Let  $(M, \omega)$  a symplectic manifold, and  $x \in M$ . Then, there exists a local chart  $(U; q^i, p_i)$  at  $x$  such that  $\omega|_U = dq^i \wedge dp_i$ .*

Some consequences of this theorem (which, in fact, predates the Poincaré theorem) are:

1. The Hamiltonian systems are locally equivalent in the following sense: given two Hamiltonian vector fields, there is a symplectic transformation that maps locally one into the other.
2. There are no local invariants in symplectic geometry (such as curvature in Riemannian geometry).
3. The group of diffeomorphisms preserving the symplectic structure is infinite dimensional.

An analogous theorem and similar consequences can be stated for presymplectic manifolds.

**Theorem 3** (Marsden-Weinstein's reduction theorem, 1974. [69]). *Let  $G$  be a Lie group which acts symplectomorphically in a symplectic manifold  $(M, \omega)$ , with associated momentum mapping  $J: M \rightarrow \mathfrak{g}^*$  (where  $\mathfrak{g}^*$  denotes the dual of the Lie algebra of  $G$ ). As  $J$  is equivariant, if  $0 \in \mathfrak{g}^*$  is a regular value of  $J$ , then  $J^{-1}(0)$  is a submanifold of  $M$  which is invariant under the action of  $G$ . Furthermore, if the action of  $G$  is free and proper, then  $J^{-1}(0)/G$  is a symplectic manifold with dimension equal to  $\dim M - 2 \dim G$ .*

The origin of this reduction procedure is very old. The original ideas come from Euler, Lagrange, Jacobi, Poisson, Lie and Noether. In modern formulation, Moser, Arnold, Guillemin, Sternberg, Marle and many others have contributed to its development and applications (see [70] and references therein).

Thus, reduction theory concerns the removal of variables using symmetries and conservation laws and, as a consequence of the theorem, every Hamiltonian system with symmetries can be reduced to another Hamiltonian system. In addition, other relevant consequences are the *Atiyah-Guillemin-Sternberg* and *Delzaut theorems* [9].

As above, this theorem can be generalized for presymplectic manifolds, as well as to other different singular cases (see, for instance, [22] and the references quoted therein).

**Theorem 4** (Kostant-Kirillov-Souriau, 1970. [51], [91]). *Let  $G$  be a Lie group, and  $\mathcal{G}$  its Lie algebra. The orbits of the coadjoint action of  $G$  in  $\mathcal{G}^*$  are endowed with a canonical symplectic structure  $\omega^* \in \Omega^2(\mathcal{G}^*)$  which is  $G$ -invariant.*

This theorem has some important corollaries:

**Corollary 1** *If the action of  $G$  on  $(M, \omega)$  is Poissonian (see [59]), then:*

1. *The associated momentum mapping  $J$  maps the orbits  $\mathcal{O}_x$  of the action of  $G$  in  $M$  into the orbits  $\mathcal{O}_{J(x)}^*$  of the coadjoint action of  $G$  in  $\mathcal{G}^*$ . Furthermore,  $J$  maps the symplectic form  $\omega$  into the natural symplectic form of  $\mathcal{O}_{J(x)}^*$ .*
2. *If the action is also transitive, then  $J$  is a local symplectomorphism.*

Roughly speaking, this means that Hamiltonian systems with symmetries are the orbits of the coadjoint action.

Another subject related to this theorem is the so-called *geometric quantization*: in a natural way, a Hilbert space  $\mathcal{H}$  can be associated with every symplectic manifold. Therefore, the phase space and the state variable functions are transformed into a Hilbert space and self-adjoint operators on it (see, for instance, [24], [50], [52], [89], [90], [102]).

**Theorem 5** (Arnold-Liouville, 1969. [2]). *Let  $(M, \omega, H)$  be a  $2n$ -dimensional integrable system with integrals of motion  $f_1 = H, f_2, \dots, f_n$ . Let  $c \in \mathbb{R}^n$  be a regular value of  $f = (f_1, \dots, f_n)$ . The corresponding level set  $f^{-1}(c)$  is a Lagrangian submanifold of  $M$ .*

1. If the flows of  $X_{f_1}, \dots, X_{f_n}$  starting at a point  $p \in f^{-1}(c)$  are complete, then the connected component of  $f^{-1}(c)$  containing  $p$  is a homogeneous space for  $\mathbb{R}^n$ . With respect to this affine structure, the component has coordinates  $\varphi_1, \dots, \varphi_n$ , known as angle coordinates, in which the flows of the vector fields  $X_{f_1}, \dots, X_{f_n}$  are linear.
2. There are coordinates  $\psi_1, \dots, \psi_n$ , which are known as action coordinates, complementary to the angle coordinates, such that the  $\psi_i$ 's are integrals of motion and  $(\varphi_1, \dots, \varphi_n, \psi_1, \dots, \psi_n)$  form a Darboux chart.

This shows that the dynamics of integrable systems (see [9]) are very simple, and they have explicit solutions in these action-angle coordinates (see [9], p.110, and [18]).

Another interesting result is the classification of normal forms of quadratic Hamiltonians (see [2], p.486), which is based on *Williamson's theorem* [101]: a result about the classification of quadratic forms in symplectic vector spaces.

We also refer to a theorem of Banyaga which states that some classical structures and, in particular, the symplectic form, are determined by their automorphism groups, the group of symplectomorphisms [4]. Closely related to this result is the *Lee Hwa Chung theorem* which establishes that, in a given symplectic manifold, apart from the symplectic and volume forms, the only differential forms invariant by the set of locally Hamiltonian vector fields are multiples of exterior powers of the symplectic form. The original version of this theorem [44] concerns the uniqueness of invariant integral forms (the *Poincaré-Cartan integral invariant*) under canonical transformations (that is, those diffeomorphisms in a symplectic manifold, mapping Hamiltonian vector fields into themselves), and this result leads to characterize canonical transformations in the Hamiltonian formalism of Mechanics, identifying them with the symplectomorphisms. Afterwards, these results were generalized to presymplectic Hamiltonian systems [28].

There are many other topics concerning symplectic geometry in the realm of Hamiltonian systems. For instance, the study of Lagrangian submanifolds and foliations, which has application to several kind of problems [99]; such as, to give a very nice interpretation of the Lagrangian and Hamiltonian dynamics and the Legendre transformation [93], [94], [95], to characterize the generating functions of canonical transformations (symplectomorphisms) [1], [99], or to get a geometrical framework for the Hamilton-Jacobi theory [97]. Apart from this, we can point out the so-called *symplectic integrators*, which allow us to obtain algorithms for the discretization of the Hamilton equations and numerical solutions of them, and are based on the conservation of the symplectic form [6], [17], [56], [67]. There is also the study of the Schrödinger equation as an infinite dimensional Hamiltonian system [96] (which is related to geometric quantization).

Finally, as presymplectic geometry is the arena for singular Hamiltonian systems, we must also mention some remarkable results concerning this topic. One of the most important is the *Weinstein extension theorem* [98], and one of its consequences, the *coisotropic imbedding theorem* [29], [62], which allow to establish the local structure of presymplectic Hamiltonian systems as systems defined in a symplectic manifold [14], [88].

## 3 Singular differential equations

### 3.1 Systems described by singular differential equations

In general, an ordinary differential equation is a relation  $\mathbf{F}(t, \mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}, \dots) = 0$  involving an independent variable  $t$ , a dependent variable  $\mathbf{x}(t)$ , and its derivatives up to a certain order. When the highest-order derivative can be solved, say  $\mathbf{x}^{(k)} = \mathbf{f}(t, \mathbf{x}, \dots)$ , the equation is said to be in normal form, which is the most appropriate for studying and solving the equation, especially thanks to the existence and uniqueness theorem. However, we are mainly interested in the case where this highest-order derivative cannot be solved, not only on a point, but on an open set, and this is what we will mean with the term “singular”.

Although of lesser importance than equations in normal form, singular differential equations have been studied for decades, especially in the last 30 years. The main reason is that they appear in many applications, under various names: singular, degenerate or implicit systems, differential-algebraic equations (DAEs), descriptor systems, ... Let us consider some of these applications.

- *Theoretical physics.* A Lagrangian  $L(t, q^i, \dot{q}^i)$  is called *singular* when its hessian matrix  $\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}$  is singular. This means that its Euler–Lagrange equations (1) are singular. Such Lagrangians appear in the description of relativistic phenomena and, as a consequence, in all the fundamental theories of physics; they were first studied by Dirac and Bergman in 1950. (See, for instance, [12]).
- *Applied mathematics.* In control theory, it is usual to deal with linear singular systems; as for instance

$$E\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}, \quad \mathbf{y} = C\mathbf{x} + D\mathbf{u},$$

where  $E$ ,  $A$ ,  $B$ ,  $C$  and  $D$  are constant matrices,  $E$  singular. Such systems appear in various problems coming from optimal control, constrained control and electric circuits, and so they have been widely studied. One of the first books on these systems was [7]; more recently [47] gives a complete account of singular linear systems. The nonlinear case has been less studied, and is related to the theory of singular perturbations [46].

- *Other applications.* Many of them can be found in circuit theory, but also in chemical and industrial engineering. We also know of applications to more faraway fields, such as econometry or biology.

It is known that the introduction of additional variables allows us to transform any differential equation into a first-order autonomous equation,  $\mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}) = 0$ , or  $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$  in normal form;. For the sake of simplicity, from now on we will consider only such equations.



## 3.2 Problems arising when solving singular systems

In general, singular systems cannot be represented by a vector field, so their integration —analytical or numerical— leads to new problems, which are not present in the regular case.

**Consistency** The first point to be noted is that a singular differential equation may not have solutions passing through each point of the phase space  $M$  of the system. In other words, not all values of the variables are admissible initial conditions.

To solve the equation, first one should identify these admissible initial conditions. Usually they are not obtained directly but in an algorithmic way. For instance, the differential equation may imply some relations  $\phi^a(\mathbf{x}) = 0$  among the variables  $\mathbf{x}$ ; these relations, sometimes called *primary constraints*, define a subset  $M_1 \subset M$ , the *primary constraint subset*. Hence, the first step consists in finding these constraints.

Nevertheless, the problem does not finish at this point: not only the initial conditions  $\mathbf{x}_o$  must be inside  $M_1$ , but also their evolution  $\mathbf{x}(t)$  must remain there —this is a tangency condition. This implies additional constraints, as will be shown later. This procedure is repeated until a final constraint subset  $M_f$  is obtained.

**Uniqueness** Under some favourable conditions of regularity, the preceding procedure ends with a submanifold, where the differential equation can be represented by a family of tangent vector fields, so that their integral curves describe the solutions. Since different vector fields yield different solutions, in general, giving  $\mathbf{x}(0) = \mathbf{x}_o$ , does not determine the solution. Sometimes, even the knowledge of all the derivatives at  $t = 0$  does not determine the solution.

This is not always the case. Sometimes the singularity of the initial differential equation arises from an “inappropriate” choice of the variables (that is, the initial state space is too large to describe the real degrees of freedom of the problem, due to the use of redundant coordinates); in such a case, the system has a unique solution on the final constraint submanifold.

**Reduction** However, in some physical problems the singularity of the system is a consequence of the existence of a certain kind of internal symmetries (called *gauge symmetry* in theoretical physics), accounting for the fact that there are different solutions representing the same physical state. In this case, the differential equation is undetermined: for every point in the final constraint submanifold which is taken as an initial condition, the evolution of the system is not determined because a multiplicity of integral curves (of different vector fields solution) pass through it.

A reduction procedure can be used to remove this ambiguity: different points of  $M_f$  that can be reached through different solutions beginning at the same initial condition must be identified. This quotient is the reduced phase space of the system. Then the physical states of the systems are identified with the points of the reduced phase space.

**Control systems** In addition to these problems, other questions are also interesting when dealing with singular control systems: controllability, observability, reachability, stabilizability, realizability, optimality ... All these concepts, which are well established for regular control systems, must be reconsidered in the singular case.

**Numerical methods** Here the matter is the design and convergence of numerical methods to solve implicit differential equations, and also their relation with singular perturbation problems. There are many articles and some books devoted to these problems (see, for instance, [8], [40], [41]).

### 3.3 Geometric formulations of singular differential equations

In geometric terms, a differential equation written in normal form is defined by a vector field  $X$  on a manifold  $M$ . Then the equation for a path  $\gamma: I \rightarrow M$  reads  $\dot{\gamma} = X \circ \gamma$ .

In the same way, an implicit differential equation can be geometrically described by a submanifold  $D \subset TM$  of the tangent bundle of  $M$ , and the differential equation is then expressed by the inclusion

$$\dot{\gamma}(I) \subset D.$$

Even if the equation is initially set in euclidian space, one is usually led to work on submanifolds of the initial space, so the geometric framework is not only nice, but also necessary.

Singular differential equations, from this most general point of view, have often been studied in the literature. For instance, in [75] and [64] this general framework is applied to the Hamiltonian dynamics of singular Lagrangian systems. The article [63] studies symmetries and constants of motion for these singular equations, whereas their integrability is studied in [74]. In the same way, but taking  $M$  as a euclidean space, [81] studies the existence of solutions of the same problem, and also gives an algorithm for finding these solutions under some regularity conditions; this article is indeed a geometric formulation of the authors' previous works [80], [83]. More or less, the same algorithm is given in [82].

In the literature, the most widespread singular differential equations are of a special type: they combine some restrictions on the base  $M$  and some linear relations among the velocities in  $TM$ . In general, these equations can be written, in coordinates, as  $A(\mathbf{x})\dot{\mathbf{x}} = b(\mathbf{x})$ , where  $A(\mathbf{x})$  is a matrix, usually singular. Such equations can be called *linearly singular* (they are also called *quasilinear*).

The geometric study of these equations has been developed independently in the areas of Theoretical Physics and Applied Mathematics.

In Theoretical Physics the initial problem was to obtain a Hamiltonian description for singular Lagrangians [19]. The geometrization of this problem was performed later [88], [60], [75], and a more general framework is that of presymplectic systems [32], where the equation of motion has the form

$$i_{\dot{\gamma}}\omega = \alpha \circ \gamma,$$

where  $\omega$  is a presymplectic form on a manifold  $M$  and  $\alpha$  is a 1-form. Hamiltonian dynamics corresponds to the case where  $M \subset T^*Q$  is a submanifold,  $\omega$  is the pullback

of the canonical symplectic form of  $T^*Q$  to  $M$ , and  $\alpha = dH$ , where  $H$  is a Hamiltonian function. An algorithm was presented in the aforementioned paper to study the existence of solutions of such a system. Note that, since the transformation  $X \mapsto i_X\omega$  is not an isomorphism, there is no guarantee of either existence or uniqueness of the solutions. The same ideas can be applied to search the solutions in the Lagrangian formalism for the singular case [30]; but then the presymplectic equation must be supplemented with a second-order condition [31], [76]. Another way to treat this problem is working in the manifold  $M = T^*Q \times_Q TQ$  [87].

In a more general way, linearly singular differential equations can be described as follows [34]: a linearly singular system is given by a vector bundle morphism  $A: TM \rightarrow F$  and a section  $b$  of the vector bundle  $F \rightarrow M$ . Then the differential equation reads

$$A \circ \dot{\gamma} = b \circ \gamma.$$

This problem can also be formulated in terms of vector fields. In this article, Dirac's algorithm for presymplectic systems [32] is generalized to linearly singular systems. This framework includes applications to other problems like singular Lagrangian formalism and higher order Lagrangians [37] and implicit Hamiltonian systems [5]. Symmetries of linearly singular systems have been studied recently [36].

A less general framework, but still including the presymplectic systems, is presented in [77], which corresponds to a linearly singular system with  $F = T^*M$ ; the constraint algorithm and symmetries are studied therein.

Furthermore, some problems in circuit theory led several researchers to consider singular differential equations.

For instance, in [92] a *constrained differential equation* is a set of data including a linear restriction on the velocities as well as some constraints on the base manifold obtained from a certain potential function. The discontinuous solutions of this problem, as well as its singularities, are analyzed.

The same geometric framework for constrained equations is presented in [39], with some simplifications. The problem is to find a curve  $c: I \rightarrow E$  such that

$$Tp \circ \dot{c} = \chi \circ c \quad c(I) \subset \Sigma,$$

where  $p: E \rightarrow B$  is a smooth map,  $\chi$  a vector field along  $p$ , and  $\Sigma$  is a submanifold of  $E$  with the same dimension as  $M$ . Notice that this problem is also linearly singular.

We can also mention the paper [16], which considers a generalized vector field, namely, a vector bundle endomorphism  $A: TM \rightarrow TM$  together with a vector field  $v$  in  $M$ . The equation of motion is then

$$A \circ \dot{\gamma} = v \circ \gamma.$$

The aim of the paper is to classify the normal forms for generalized vector fields. Later papers have studied normal forms [73] and stability [84] of such equations.

Finally, a brief review on singular system of differential equations is presented in [33], where their geometric features, including a geometric method for obtaining a non-numerical solution, are analyzed.

### 3.4 Geometric solution of singular equations

From a geometric viewpoint, the search for the solutions of an implicit equation given by a submanifold  $D \subset TM$  can be sketched very clearly: since a solution  $\gamma$  satisfies  $\dot{\gamma}(t) \in D$ , necessarily  $\gamma(t)$  belongs to  $M_1 = \tau(D)$ , where  $\tau: TM \rightarrow M$  is the natural projection. If this is a submanifold (the primary constraint submanifold), we obtain a new implicit equation, defined by the subset  $D_1 \subset TM_1$ , where  $D_1 = D \cap TM_1$ . This is the first step of an algorithm that, assuming appropriate regularity conditions, may lead to a consistent differential equation on a certain submanifold  $M_f$ .

The same algorithm applies to a linearly singular system  $(A: TM \rightarrow F, b)$ , where we want to solve the implicit equation  $A \circ \dot{\gamma} = b \circ \gamma$ . In this case, the equation  $A_x \cdot X(x) = \sigma(x)$  for the unknown vector  $X(x)$  can be solved only at the points  $x \in M$  where this linear equation is consistent:  $b(x) \in \text{Im } A_x$ . These points constitute the set  $M_1$ . Of course, the solutions must be contained in  $M_1$ , so it is clear that they are also solutions of the linearly singular system  $(A_1: TM_1 \rightarrow F_1, b_1)$  obtained by restricting  $F$ ,  $A$  and  $b$  to  $M_1$ . This is essentially the first step of the constraint algorithm in Dirac's theory. Again, under appropriate regularity conditions, this algorithm may lead to a consistent linearly differential system on a final constraint submanifold  $M_f$ , where  $b_f(x) \in \text{Im } A_{fx}$ . Then the solutions of the differential equation correspond to the integral curves of a family of vector fields  $X_f + \text{Ker } A_f$ . Finally, let us remark that this algorithm may be presented explicitly in terms of the constraints and vector fields in a suitable way for computation.

## 4 Field theory and multisymplectic geometry

### 4.1 Geometric formulation of field theory

As one may see in Table 1, many of the main PDEs in Physics can be obtained from a Lagrangian function. There are other interesting PDEs describing dynamical processes, which are also variational with other conditions: constrained problems, higher-order Lagrangians, etc. In any case, in all of them, the underlying geometric problems have the same level of difficulty as those we are going to analyze next.

The geometric way of describing these problems, for first-order theories (see, for instance, [20], [25], [26], [27], [49], [86]), consists in considering a fibered manifold  $\pi: Y \rightarrow X$  (where  $X$  is an  $m$ -dimensional oriented manifold, with volume form  $\omega \in \Omega^m(X)$ ), and the manifold of first-order jets of sections,  $J^1Y \xrightarrow{\pi^1} Y \xrightarrow{\pi} X$ . In this situation, there are natural geometric structures, such as: the vertical subbundle, the vertical endomorphisms, the canonical structure form, the module of total derivations, . . . . Using some of them, we can associate to a Lagrangian function  $L: J^1Y \rightarrow \mathbb{R}$  a closed form  $\Omega_L \in \Omega^{m+1}(J^1Y)$  such that a section  $\psi: X \rightarrow Y$  is *critical* for the variational problem  $S[\psi] = \int_M (j^1\psi)^*(L\omega)$  if, and only if,

$$(j^1\psi)^* i_Z \Omega_L = 0 \quad , \quad \text{for every } Z \in \mathfrak{X}(J^1Y)$$

This equation can be written in other equivalent ways, using  $m$ -vector fields or Ehresmann connections [21], [54]. In a local chart of adapted natural coordinates in  $J^1Y$ ,

Name	Equation	Lagrangian
Dirac	$\gamma^i \frac{\partial \psi}{\partial x^i} = m \mathbf{I} \psi$ $\gamma^i$ : Dirac matrices $\mathbf{I}$ : identity matrix	$\bar{\psi} \gamma^i \frac{\partial \psi}{\partial x^i} - \bar{\psi} \psi$
Einstein (in vacuum)	$\mathcal{R}_{\mu\nu}(g) = 0$ $g$ : metric ; $\mathcal{R}$ : Ricci tensor	$R(g) \sqrt{-\det g}$ $R$ : curvature
Elasticity	$\frac{\partial^2 u^\alpha}{\partial t^2} = \frac{\partial}{\partial x^i} \left( \frac{\partial W}{\partial u_i^\alpha} \right)$ $W(\mathbf{x}, \nabla u)$ : stored energy	$\frac{1}{2}  u_t ^2 - W$
Fluid dynamics	$\frac{\partial \mathbf{u}}{\partial t} + \nabla_{\mathbf{u}} \mathbf{u} = -\text{grad } p$ $\text{div } \mathbf{u} = 0$ $\mathbf{u}$ : fluid velocity; $p$ : pressure	$\frac{1}{2} \langle \mathbf{u}, \mathbf{u} \rangle$
Klein-Gordon	$\frac{\partial^2 \psi}{\partial t^2} = \nabla^2 \psi - m^2 \psi$	$\frac{1}{2} \left[ \left( \frac{\partial \psi}{\partial t} \right)^2 - (\nabla \psi)^2 - m^2 \psi^2 \right]$
Korteweg-de Vries	$\frac{\partial u}{\partial t} - 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$	$\frac{1}{2} \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial t} - \left( \frac{\partial \phi}{\partial x} \right)^3 - \frac{1}{2} \frac{\partial^2 \phi}{\partial x^2}$ with $u = \frac{\partial \phi}{\partial x}$
Laplace	$\nabla^2 \psi = 0$	$\frac{1}{2}  \nabla \psi ^2$
Maxwell	$\text{div } \mathbf{E} = \rho$ , $\text{curl } \mathbf{H} = \frac{\partial \mathbf{E}}{\partial t} + \mathbf{j}$ $\text{div } \mathbf{H} = 0$ , $\text{curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ $\mathbf{E}, \mathbf{H}$ : electric, magnetic fields $\rho$ : electric charge $\mathbf{j}$ : electric current	$\frac{E^2 - B^2}{8\pi} - \rho \phi + \mathbf{j} \cdot \mathbf{A}$ with $\mathbf{E} = -\text{grad } \phi - \frac{\partial \mathbf{A}}{\partial t}$ $\mathbf{B} = \text{curl } \mathbf{A}$ $\phi, \mathbf{A}$ : electric, magnetic potentials
Schrödinger	$-\frac{1}{2m} \nabla^2 \psi + V \psi = i \frac{\partial \psi}{\partial t}$	$\sqrt{-1} \bar{\psi} \psi - \frac{1}{2m} g^{ij} \frac{\partial \psi}{\partial x^i} \frac{\partial \psi}{\partial x^j} - V(x^i, t) \bar{\psi} \psi$ $g$ : metric
Sine-Gordon	$\frac{\partial^2 \psi}{\partial x \partial t} = \sin \psi$	$\frac{1}{2} \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial t} - \cos \psi$
Wave	$\frac{\partial^2 \phi}{\partial t^2} = \nabla^2 \phi$	$\frac{1}{2} (\dot{\phi})^2 - \ \nabla \phi\ ^2$

Table 1: Some of the main PDEs in physics, with their corresponding Lagrangian.

$(x^\alpha, y^A, v_\alpha^A)$ , the expression of this condition is the well known system of Euler-Lagrange equations for the multiple variational case:

$$\frac{\partial L}{\partial y^A} \Big|_{j^1\psi} - \frac{\partial}{\partial x^\alpha} \left( \frac{\partial L}{\partial v_\alpha^A} \Big|_{j^1\psi} \right) = 0 \quad ; \quad A = 1, \dots, N$$

It is interesting to point out that, in the particular case with  $\dim X = 1$ , this framework describes the time-dependent mechanics [55], [61], and then the multisymplectic form reduces to be a *contact* or a *cosymplectic form*.

As in the case of ODEs, a Hamiltonian formulation is also possible in this case [13], [23], [42], [68], [79], [85], after defining the corresponding Legendre map. In all of them, the so-called *multimomentum phase space* is a fiber bundle over  $Y$  endowed with a multisymplectic structure, which is canonical in some cases, although in others it is constructed using additional elements (connections or sections in the bundle). The PDEs obtained in these formalisms are called the *Hamilton-De Donder-Weyl equations*.

To sum up, in all these formulations the underlying geometric structure is a couple  $(M, \Omega)$ , where  $M$  is a differentiable manifold and  $\Omega$  is a closed  $k$ -form ( $k > 2$ ), and is called a *multisymplectic manifold* if  $\Omega$  is 1-nondegenerate.

Natural examples of this structure are  $\mathbb{R}^n$  and  $S^n$  ( $n > 2$ ) with their volume elements. Canonical models of multisymplectic manifolds are a generalization of the cotangent bundle of a manifold  $Q$ : the multicotangent bundle  $\Lambda^k T^*Q$  ( $1 < k \leq \dim Q$ ); that is, the bundle of exterior  $k$ -forms on  $Q$ , and the subbundles of it (bundles of forms). All of them are endowed with a natural multisymplectic  $(k+1)$ -form (i.e.; closed and 1-nondegenerate). Other examples are semisimple Lie groups, cosymplectic manifolds and Calabi-Yau manifolds (see [45]).

Finally, it must be remarked that multisymplectic geometry is not in fact the only geometrical framework for describing these kinds of variational systems of PDEs, and alternative, sometimes equivalent, geometric structures are possible: the so-called *polysymplectic* [38], [48], *k-symplectic* [3], [78], *k-cosymplectic* [58], and *Lepagean* [53] formalisms.

## 4.2 Results and open problems on multisymplectic geometry

Starting from a multisymplectic manifold  $(M, \Omega)$ , one can expect to obtain results concerning the geometric structure of  $M$  and the PDEs defined on  $M$ , in an analogous way, as in the case of symplectic geometry. Research on these topics has just started (see, for instance, [10], [11], [45]), and many of those problems are still unsolved. Next we review some of these problems and their current status.

The first fundamental result in symplectic geometry was the Darboux theorem, which stated that all symplectic manifolds are locally isomorphic. This result also holds in some particular cases of multisymplectic geometry; for instance, when the degree of  $\Omega$  is equal to  $\dim M$  (volume forms). Nevertheless, in the general case, a multisymplectic manifold does not admit a system of Darboux coordinates for the multisymplectic form [71]. In fact this is a problem arising from linear algebra: the classification of skew-symmetric tensors of degree greater than two is still an open problem. The kind of multisymplectic manifolds admitting Darboux coordinates has been identified recently [11], [57], and they are those being locally multisymplectomorphic to bundles of forms.

Nevertheless, further developments have not been achieved. For instance, concerning reduction theory, only partial results about reduction by foliations are currently being studied [45]. The theory of reduction of systems of “multisymplectic” PDEs under the action of groups of symmetries, obtaining other simpler but also “multisymplectic” ones, is under research [15], [43], [70].

In the same way, approaches for generalizing symplectic integrators to this geometric framework (i.e., the so-called *multisymplectic integrators*) have only been recently developed [66].

Other results similar to those stated in Theorems 4 (Konstant-Kirillov-Souriau) and 5 (Arnold-Liouville) have not been achieved yet.

One can expect to see more work on all these subjects in the future.

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